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RAMSEY NUMBERS FOR TREES

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ABSTRACT. For $n \geq 5$ let T'_n denote the unique tree on n vertices with $\Delta(T'_n) = n - 2$, and let $T_n^* = (V, E)$ be the tree on n vertices with $V = \{v_0, v_1, \dots, v_{n-1}\}$ and $E = \{v_0v_1, \dots, v_0v_{n-3}, v_{n-3}v_{n-2}, v_{n-2}v_{n-1}\}$. In this paper we evaluate the Ramsey numbers $r(G_m, T'_n)$ and $r(G_m, T_n^*)$, where G_m is a connected graph of order m . As examples, for $n \geq 8$ we have $r(T'_n, T_n^*) = r(T_n^*, T_n^*) = 2n - 5$, for $n > m \geq 7$ we have $r(K_{1,m-1}, T_n^*) = m + n - 3$ or $m + n - 4$ according as $m - 1 \mid (n - 3)$ or $m - 1 \nmid (n - 3)$, for $m \geq 7$ and $n \geq (m - 3)^2 + 2$ we have $r(T_m^*, T_n^*) = m + n - 3$ or $m + n - 4$ according as $m - 1 \mid (n - 3)$ or $m - 1 \nmid (n - 3)$.

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1. Introduction.

In this paper, all graphs are simple graphs. For a graph $G = (V(G), E(G))$ let $e(G) = |E(G)|$ be the number of edges in G and let $\Delta(G)$ be the maximal degree of G . For a forbidden graph L , let $ex(p; L)$ denote the maximal number of edges in a graph of order p not containing L as a subgraph. The corresponding Turán's problem is to evaluate $ex(p; L)$.

Let \mathbb{N} be the set of positive integers, and let $p, n \in \mathbb{N}$ with $p \geq n \geq 3$. For a given tree T_n on n vertices, it is difficult to determine the value of $ex(p; T_n)$. The famous Erdős-Sós conjecture asserts that $ex(p; T_n) \leq \frac{(n-2)p}{2}$ for every tree T_n on n vertices. For the progress on the Erdős-Sós conjecture, see [4, 8, 9, 11]. Write $p = k(n - 1) + r$, where $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n - 2\}$. Let P_n be the path on n vertices. In [5] Faudree and Schelp showed that

$$(1.1) \quad ex(p; P_n) = k \binom{n-1}{2} + \binom{r}{2} = \frac{(n-2)p - r(n-1-r)}{2}.$$

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In the special case $r = 0$, (1.1) is due to Erdős and Gallai [3]. Let $K_{1,n-1}$ denote the unique tree on n vertices with $\Delta(K_{1,n-1}) = n - 1$, and for $n \geq 4$ let T'_n denote the unique tree on n vertices with $\Delta(T'_n) = n - 2$. In [10] the author and Lin-Lin Wang obtained exact values of $ex(p; K_{1,n-1})$ and $ex(p; T'_n)$, see Lemmas 2.4 and 2.5.

For $n \geq 5$ let $T_n^* = (V, E)$ be the tree on n vertices with $V = \{v_0, v_1, \dots, v_{n-1}\}$ and $E = \{v_0v_1, \dots, v_0v_{n-3}, v_{n-3}v_{n-2}, v_{n-2}v_{n-1}\}$. In [10], we also determine the value of $ex(p; T_n^*)$, see Lemmas 2.6-2.8.

As usual \overline{G} denotes the complement of a graph G . Let G_1 and G_2 be two graphs. The Ramsey number $r(G_1, G_2)$ is the smallest positive integer n such that, for every graph G with n vertices, either G contains a copy of G_1 or else \overline{G} contains a copy of G_2 .

Let $n \in \mathbb{N}$ with $n \geq 6$. If the Erdős-Sós conjecture is true, it is known that $r(T_n, T_n) \leq 2n - 2$ (see [8]). Let $m, n \in \mathbb{N}$. In 1973 Burr and Roberts[2] showed that for $m, n \geq 3$,

$$(1.2) \quad r(K_{1,m-1}, K_{1,n-1}) = \begin{cases} m + n - 3 & \text{if } 2 \nmid mn, \\ m + n - 2 & \text{if } 2 \mid mn. \end{cases}$$

In 1995, Guo and Volkmann[6] proved that for $n \geq m \geq 5$,

$$(1.3) \quad r(T'_m, T'_n) = \begin{cases} m + n - 3 & \text{if } m - 1 \mid (n - 3), \\ m + n - 5 & \text{if } m = n \equiv 0 \pmod{2}, \\ m + n - 4 & \text{otherwise} \end{cases}$$

and for $n > m \geq 4$,

$$(1.4) \quad r(K_{1,m-1}, T'_n) = \begin{cases} m + n - 3 & \text{if } 2 \mid m(n - 1), \\ m + n - 4 & \text{if } 2 \nmid m(n - 1). \end{cases}$$

Let $m, n \in \mathbb{N}$ with $n \geq m \geq 6$. In this paper we evaluate the Ramsey number $r(T_m, T_n^*)$ for $T_m \in \{P_m, K_{1,m-1}, T'_m, T_m^*\}$. As examples, for $n \geq 8$ we have

$$(1.5) \quad r(P_n, T_n^*) = r(T_n^*, T_n^*) = 2n - 5,$$

for $n > m \geq 7$ we have

$$(1.6) \quad r(K_{1,m-1}, T_n^*) = \begin{cases} m + n - 3 & \text{if } m - 1 \mid (n - 3), \\ m + n - 4 & \text{if } m - 1 \nmid (n - 3), \end{cases}$$

for $m \geq 7$ and $n \geq (m - 3)^2 + 2$ we have

$$(1.7) \quad r(P_m, T_n^*) = r(T'_m, T_n^*) = r(T_m^*, T_n^*) = \begin{cases} m + n - 3 & \text{if } m - 1 \mid (n - 3), \\ m + n - 4 & \text{if } m - 1 \nmid (n - 3). \end{cases}$$

In addition to the above notation, throughout the paper we also use the following notation: $\lfloor x \rfloor$ —the greatest integer not exceeding x , K_n —the complete graph on n vertices, $K_{m,n}$ —the complete bipartite graph with m and n vertices in the bipartition, $d_G(v)$ —the degree of the vertex v in given graph G , $d(u, v)$ —the distance between the two vertices u and v in a graph.

2. Basic lemmas.

Lemma 2.1. *Let G_1 and G_2 be two graphs. Suppose $p \in \mathbb{N}, p \geq \max\{|V(G_1)|, |V(G_2)|\}$ and $ex(p; G_1) + ex(p; G_2) < \binom{p}{2}$. Then $r(G_1, G_2) \leq p$.*

Proof. Let G be a graph of order p . If $e(G) \leq ex(p; G_1)$ and $e(\overline{G}) \leq ex(p; G_2)$, then

$$ex(p; G_1) + ex(p; G_2) \geq e(G) + e(\overline{G}) = \binom{p}{2}.$$

This contradicts the assumption. Hence, either $e(G) > ex(p; G_1)$ or $e(\overline{G}) > ex(p; G_2)$. Therefore, G contains a copy of G_1 or \overline{G} contains a copy of G_2 . This shows that $r(G_1, G_2) \leq |V(G)| = p$. So the lemma is proved.

Lemma 2.2. *Let $k, p \in \mathbb{N}$ with $p \geq k + 1$. Then there exists a k -regular graph of order p if and only if $2 \mid kp$.*

This is a known result, see for example [10, Corollary 2.1].

Lemma 2.3. *Let G_1 and G_2 be two graphs with $\Delta(G_1) = d_1 \geq 2$ and $\Delta(G_2) = d_2 \geq 2$. Then*

- (i) $r(G_1, G_2) \geq d_1 + d_2 - (1 - (-1)^{(d_1-1)(d_2-1)})/2$.
- (ii) *Suppose that G_1 is a connected graph of order m and $d_1 < d_2 \leq m$. Then $r(G_1, G_2) \geq 2d_2 - 1 \geq d_1 + d_2$.*
- (iii) *Suppose that G_1 is a connected graph of order m and $d_2 > m$. If one of the conditions*

$$(1) \ 2 \mid (d_1 + d_2 - m),$$

$$(2) \ d_1 \neq m - 1,$$

$$(3) \ G_2 \text{ has two vertices } u \text{ and } v \text{ such that } d(v) = \Delta(G_2) \text{ and } d(u, v) = 3$$

holds, then $r(G_1, G_2) \geq d_1 + d_2$.

Proof. We first consider (i). If $2 \mid (d_1 - 1)(d_2 - 1)$, then $2 \mid (d_1 - 1)(d_1 + d_2 - 1)$. Since $d_1 - 1 \geq 1$, by Lemma 2.2 we may construct a $d_1 - 1$ -regular graph G of order $d_1 + d_2 - 1$. Since $\Delta(G) = d_1 - 1$ and $\Delta(\overline{G}) = d_2 - 1$, G does not contain G_1 as a subgraph and \overline{G} does not contain G_2 as a subgraph. Hence $r(G_1, G_2) \geq 1 + |V(G)| = d_1 + d_2$. Now we assume $2 \nmid (d_1 - 1)(d_2 - 1)$. Then $2 \mid d_1$, $2 \mid d_2$ and so $2 \mid (d_1 + d_2 - 2)$. By Lemma 2.2, we may construct a $d_1 - 1$ -regular graph G of order $d_1 + d_2 - 2$. Since $\Delta(G) = d_1 - 1$ and $\Delta(\overline{G}) = d_2 - 2$, G does not contain G_1 as a subgraph and \overline{G} does not contain G_2 as a subgraph. Hence $r(G_1, G_2) \geq 1 + |V(G)| = d_1 + d_2 - 1$. This proves (i).

Next we consider (ii). Suppose that G_1 is a connected graph of order m and $d_1 < d_2 \leq m$. Since $K_{d_2-1} \cup K_{d_2-1}$ does not contain any copies of G_1 , and its complement K_{d_2-1, d_2-1} does not contain any copies of G_2 , we see that $r(G_1, G_2) \geq 1 + 2(d_2 - 1) = 2d_2 - 1 \geq d_1 + d_2$. This proves (ii).

Finally we consider (iii). Suppose that G_1 is a connected graph of order m and $d_2 > m$. By Lemma 2.2, we may construct a graph

$$G = \begin{cases} K_{m-1} \cup H_1 & \text{if } 2 \mid (d_1 + d_2 - m), \\ K_{m-2} \cup H_2 & \text{if } 2 \nmid (d_1 + d_2 - m), \end{cases}$$

where H_1 is a $d_1 - 1$ -regular graph of order $d_1 + d_2 - m$ and H_2 is a $d_1 - 1$ -regular graph of order $d_1 + d_2 - m + 1$. It is easily seen that G does not contain any copies of G_1 and

$$\Delta(\overline{G}) = \begin{cases} d_2 - 1 & \text{if } 2 \mid (d_1 + d_2 - m) \text{ or } d_1 \neq m - 1, \\ d_2 & \text{if } 2 \nmid (d_1 + d_2 - m) \text{ and } d_1 = m - 1. \end{cases}$$

If $2 \mid (d_1 + d_2 - m)$ or $d_1 \neq m - 1$, then \overline{G} does not contain any copies of G_2 and so $r(G_1, G_2) \geq 1 + |V(G)| = d_1 + d_2$. Now assume $2 \nmid (d_1 + d_2 - m)$ and $d_1 = m - 1$. For $v_0 \in V(H_2)$ we have $d_{\overline{G}}(v_0) = d_2 - 1$. Suppose that $v_1, \dots, v_{m-2} \in V(G)$ and v_1, \dots, v_{m-2} induce a copy of K_{m-2} . Then $\{v_1, \dots, v_{m-2}\}$ is an independent set in \overline{G} and $d_{\overline{G}}(v_i) = d_2$ for $i = 1, 2, \dots, m - 2$. If G_2 has two vertices u and v such that $d(v) = \Delta(G_2)$ and $d(u, v) = 3$, we see that G does not contain any copies of G_2 and so $r(G_1, G_2) \geq 1 + |V(G)| = d_1 + d_2$. This proves (iii) and the lemma is proved.

Lemma 2.4 ([10, Theorem 2.1]). *Let $p, n \in \mathbb{N}$ with $p \geq n - 1 \geq 1$. Then $ex(p; K_{1, n-1}) = \lfloor \frac{(n-2)p}{2} \rfloor$.*

Lemma 2.5 ([10, Theorem 3.1]). *Let $p, n \in \mathbb{N}$ with $p \geq n \geq 5$. Let $r \in \{0, 1, \dots, n - 2\}$ be given by $p \equiv r \pmod{n - 1}$. Then*

$$ex(p; T'_n) = \begin{cases} \lfloor \frac{(n-2)(p-1)-r-1}{2} \rfloor & \text{if } n \geq 7 \text{ and } 2 \leq r \leq n - 4, \\ \frac{(n-2)p-r(n-1-r)}{2} & \text{otherwise.} \end{cases}$$

Lemma 2.6 ([10, Theorems 4.1-4.3]). *Let $p, n \in \mathbb{N}$ with $p \geq n \geq 6$, and let $p = k(n - 1) + r$ with $k \in \mathbb{N}$ and $r \in \{0, 1, n - 5, n - 4, n - 3, n - 2\}$. Then*

$$ex(p; T_n^*) = \begin{cases} \frac{(n-2)(p-2)}{2} + 1 & \text{if } n > 6 \text{ and } r = n - 5, \\ \frac{(n-2)p-r(n-1-r)}{2} & \text{otherwise.} \end{cases}$$

Lemma 2.7 ([10, Theorem 4.4]). *Let $p, n \in \mathbb{N}$, $p \geq n \geq 11$, $r \in \{2, 3, \dots, n - 6\}$ and $p \equiv r \pmod{n - 1}$. Let $t \in \{0, 1, \dots, r + 1\}$ be given by $n - 3 \equiv t \pmod{r + 2}$. Then*

$$ex(p; T_n^*) = \begin{cases} \lfloor \frac{(n-2)(p-1)-2r-t-3}{2} \rfloor & \text{if } r \geq 4 \text{ and } 2 \leq t \leq r - 1, \\ \frac{(n-2)(p-1)-t(r+2-t)-r-1}{2} & \text{otherwise.} \end{cases}$$

Lemma 2.8 ([10, Theorem 4.5]). *Let $p, n \in \mathbb{N}$ with $6 \leq n \leq 10$ and $p \geq n$, and let $r \in \{0, 1, \dots, n - 2\}$ be given by $p \equiv r \pmod{n - 1}$.*

- (i) *If $n = 6, 7$, then $ex(p; T_n^*) = \frac{(n-2)p-r(n-1-r)}{2}$.*
- (ii) *If $n = 8, 9$, then*

$$ex(p; T_n^*) = \begin{cases} \frac{(n-2)p-r(n-1-r)}{2} & \text{if } r \neq n - 5, \\ \frac{(n-2)(p-2)}{2} + 1 & \text{if } r = n - 5. \end{cases}$$

4

(iii) If $n = 10$, then

$$ex(p; T_n^*) = \begin{cases} 4p - \frac{r(9-r)}{2} & \text{if } r \neq 4, 5, \\ 4p - 7 & \text{if } r = 5, \\ 4p - 9 & \text{if } r = 4. \end{cases}$$

Lemma 2.9. Let $p, m \in \mathbb{N}$ with $p \geq m \geq 5$, and $T_m \in \{P_m, K_{1,m-1}, T'_m, T_m^*\}$. Then $ex(p; T_m) \leq \frac{(m-2)p}{2}$. Moreover, if $m-1 \nmid p$ and $T_m \in \{P_m, T'_m, T_m^*\}$, then $ex(p; T_m) \leq \frac{(m-2)(p-1)}{2}$.

Proof. This is immediate from (1.1) and Lemmas 2.4-2.8.

Lemma 2.10. Let $m, n \in \mathbb{N}$ with $m, n \geq 5$. Let G_m be a connected graph on m vertices. If $m+n-5 = (m-1)x + (m-2)y$ for some nonnegative integers x and y , then $r(G_m, T_n) \geq m+n-4$ for $T_n \in \{K_{1,n-1}, T'_n, T_n^*\}$.

Proof. Let $G = xK_{m-1} \cup yK_{m-2}$. Then $|V(G)| = m+n-5$, $\Delta(G) \leq m-1$ and $\Delta(\overline{G}) \leq n-3$. Clearly, G does not contain G_m as a subgraph, and \overline{G} does not contain T_n as a subgraph. So the result is true.

Lemma 2.11 ([7, Theorem 8.3, pp.11-12]). Let $a, b, n \in \mathbb{N}$. If a is coprime to b and $n \geq (a-1)(b-1)$, then there are two nonnegative integers x and y such that $n = ax + by$.

Conjecture 2.1. Let $p, n \in \mathbb{N}$, $p \geq n \geq 5$, $p = k(n-1) + r$, $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n-2\}$. Let $T_n \neq K_{1,n-1}, T'_n$ be a tree on n vertices. Then $ex(p; T_n) \leq ex(p; T_n^*)$. Hence

(i) If $r \in \{0, 1, n-4, n-3, n-2\}$, then

$$ex(p; T_n) = \frac{(n-2)p - r(n-1-r)}{2}.$$

(ii) If $2 \leq r \leq n-5$, then

$$ex(p; T_n) \leq \frac{(n-2)(p-1) - r - 1}{2}.$$

We note that

$$(2.1) \quad ex(p; T_n) \geq e(kK_{n-1} \cup K_r) = \frac{(n-2)p - r(n-1-r)}{2} = ex(p; P_n).$$

Definition 2.1. For $n \geq 5$ let T_n be a tree on n vertices. View T_n as a bipartite graph with s_1 and s_2 vertices in the bipartition. Define $\alpha_2(T_n) = \max \{s_1, s_2\}$.

Conjecture 2.2. Let $p, n \in \mathbb{N}$ with $p \geq n \geq 5$. Let $T_n^{(1)}$ and $T_n^{(2)}$ be two trees on n vertices. If $\alpha_2(T_n^{(1)}) < \alpha_2(T_n^{(2)})$, then $ex(p; T_n^{(1)}) \leq ex(p; T_n^{(2)})$.

3. The Ramsey number $r(G_n, T_n^*)$.

Lemma 3.1. *Let $n \in \mathbb{N}$, $n \geq 6$, and let G_n be a connected graph on n vertices such that $ex(2n-5; G_n) < n^2 - 5n + 4$. Then $r(G_n, T_n^*) = 2n - 5$.*

Proof. As $2K_{n-3}$ does not contain any copies of G_n and $\overline{2K_{n-3}} = K_{n-3, n-3}$ does not contain any copies of T_n^* , we see that $r(G_n, T_n^*) > 2(n-3)$. By Lemma 2.6 we have

$$ex(2n-5; T_n^*) = \frac{(n-2)(2n-5) - 3(n-4)}{2} = n^2 - 6n + 11.$$

Thus,

$$\begin{aligned} ex(2n-5; G_n) + ex(2n-5; T_n^*) &< n^2 - 5n + 4 + n^2 - 6n + 11 \\ &= 2n^2 - 11n + 15 = \binom{2n-5}{2}. \end{aligned}$$

Appealing to Lemma 2.1 we obtain $r(G_n, T_n^*) \leq 2n - 5$. So $r(G_n, T_n^*) = 2n - 5$ as asserted.

Theorem 3.1. *Let $n \in \mathbb{N}$ with $n \geq 8$. Then*

$$r(P_n, T_n^*) = r(T'_n, T_n^*) = r(T_n^*, T_n^*) = 2n - 5.$$

Proof. By Lemma 2.6 we have

$$ex(2n-5; T_n^*) = \frac{(n-2)(2n-5) - 3(n-4)}{2} = n^2 - 6n + 11 < n^2 - 5n + 4.$$

By Lemma 2.5 we have

$$\begin{aligned} ex(2n-5; T'_n) &= \left\lfloor \frac{(n-2)(2n-6) - (n-4) - 1}{2} \right\rfloor = \left\lfloor n^2 - \frac{11}{2}n + \frac{15}{2} \right\rfloor \\ &\leq n^2 - \frac{11}{2}n + \frac{15}{2} < n^2 - 5n + 4. \end{aligned}$$

By (1.1) we have

$$ex(2n-5; P_n) = \binom{n-1}{2} + \binom{n-4}{2} = n^2 - 6n + 11 < n^2 - 5n + 4.$$

Thus applying Lemma 3.1 we deduce the result.

Conjecture 3.1. *Let $n \in \mathbb{N}$, $n \geq 8$, and let $T_n \neq K_{1, n-1}$ be a tree on n vertices. Then $r(T_n, T_n^*) = 2n - 5$.*

Remark 3.1 Let $n \in \mathbb{N}$ with $n \geq 4$. From [6, Theorem 3.1(ii)] we know that $r(K_{1, n-1}, T_n^*) = 2n - 3$.

4. The Ramsey number $r(G_m, T_n^*)$ for $m < n$.

Theorem 4.1. *Let $m, n \in \mathbb{N}$, $n > m \geq 5$ and $m - 1 \mid (n - 3)$. Let G_m be a connected graph of order m such that $ex(m + n - 3; G_m) \leq \frac{(m-2)(m+n-3)}{2}$ or $G_m \in \{P_m, K_{1,m-1}, T'_m, T_m^*\}$. Then $r(G_m, T_n^*) = m + n - 3$.*

Proof. By Lemma 2.9 we may assume $ex(m + n - 3; G_m) \leq \frac{(m-2)(m+n-3)}{2}$. Suppose $n - 3 = k(m - 1)$. Clearly $(k + 1)K_{m-1}$ does not contain G_m as a subgraph and $(k + 1)K_{m-1}$ does not contain T_n^* as a subgraph. Thus

$$r(G_m, T_n^*) > (k + 1)(m - 1) = m + n - 4.$$

Since $1 \leq m - 4 \leq n - 6$, using Lemma 2.9 we see that

$$ex(m + n - 3; T_n^*) \leq \frac{(n - 2)(m + n - 4)}{2}.$$

Thus,

$$\begin{aligned} & ex(m + n - 3; G_m) + ex(m + n - 3; T_n^*) \\ & \leq \frac{(m - 2)(m + n - 3)}{2} + \frac{(n - 2)(m + n - 4)}{2} \\ & < \frac{(m - 2 + n - 2)(m + n - 3)}{2} = \binom{m + n - 3}{2}. \end{aligned}$$

Hence, by Lemma 2.1 we have $r(G_m, T_n^*) \leq m + n - 3$. So the result follows.

Lemma 4.1. *Let $m, n \in \mathbb{N}$, $n > m \geq 7$ and $m - 1 \nmid (n - 3)$. Let G_m be a connected graph of order m such that $ex(m + n - 4; G_m) \leq \frac{(m-2)(m+n-4)}{2}$ or $G_m \in \{P_m, K_{1,m-1}, T'_m, T_m^*\}$. Then $r(G_m, T_n^*) \leq m + n - 4$.*

Proof. By Lemma 2.9 we may assume $ex(m + n - 4; G_m) \leq \frac{(m-2)(m+n-4)}{2}$. As $m + n - 4 = n - 1 + m - 3$ and $m - 1 \nmid (n - 3)$, we see that $2 \leq m - 3 \leq n - 4$ and $m - 3 \neq n - 5$. Thus applying Lemmas 2.6, 2.7 and 2.8 we find

$$ex(m + n - 4; T_n^*) < \frac{(n - 3)(m + n - 4)}{2}.$$

Hence,

$$\begin{aligned} & ex(m + n - 4; G_m) + ex(m + n - 4; T_n^*) \\ & < \frac{(m - 2)(m + n - 4)}{2} + \frac{(n - 3)(m + n - 4)}{2} = \binom{m + n - 4}{2}. \end{aligned}$$

Now applying Lemma 2.1 we obtain the result.

Theorem 4.2. *Let $m, n \in \mathbb{N}, n > m \geq 7$ and $m-1 \nmid (n-3)$. Let G_m be a connected graph of order m such that $ex(m+n-4; G_m) \leq \frac{(m-2)(m+n-4)}{2}$ or $G_m \in \{P_m, T'_m, T_m^*\}$. If $m+n-5 = (m-1)x + (m-2)y$ for some $x, y \in \{0, 1, 2, \dots\}$, then $r(G_m, T_n^*) = m+n-4$.*

Proof. By Lemma 4.1, $r(G_m, T_n^*) \leq m+n-4$. By Lemma 2.10, $r(G_m, T_n^*) \geq m+n-4$. Thus the result follows.

Theorem 4.3. *Suppose $m, n \in \mathbb{N}, n > m \geq 7, n = k(m-1) + b = q(m-2) + a, k, q \in \mathbb{N}, a \in \{0, 1, \dots, m-3\}$ and $b \in \{0, 1, \dots, m-2\} - \{3\}$. Let G_m be a connected graph of order m such that $ex(m+n-4; G_m) \leq \frac{(m-2)(m+n-4)}{2}$ or $G_m \in \{P_m, T'_m, T_m^*\}$. If one of the following conditions*

- (i) $b \in \{1, 2, 4\}$,
- (ii) $b = 0$ and $k \geq 3$,
- (iii) $n \geq (m-3)^2 + 2$,
- (iv) $n \geq m^2 - 1 - b(m-2)$,
- (v) $a \geq 3$ and $n \geq (a-4)(m-1) + 4$

holds, then $r(G_m, T_n^) = m+n-4$.*

Proof. For $b \in \{1, 2, 4\}$ we have

$$m+n-5 = \begin{cases} (k-2)(m-1) + 3(m-2) & \text{if } b = 1, \\ (k-1)(m-1) + 2(m-2) & \text{if } b = 2, \\ (k+1)(m-1) & \text{if } b = 4. \end{cases}$$

For $b = 0$ and $k \geq 3$ we have $m+n-5 = (k-3)(m-1) + 4(m-2)$. For $n \geq (m-3)^2 + 2$, we have $m+n-5 \geq (m-2)(m-3)$ and so $m+n-5 = (m-1)x + (m-2)y$ for some $x, y \in \{0, 1, 2, \dots\}$ by Lemma 2.11. For $n \geq m^2 - 1 - b(m-2)$ we have $k \geq m+1-b$ and $m+n-5 = (k+b-m-1)(m-1) + (m+3-b)(m-2)$. For $a \geq 3$ and $n \geq (a-4)(m-1) + 4$ we have $q \geq a-4$ and $m+n-5 = (a-3)(m-1) + (q+4-a)(m-2)$. Now combining all the above with Theorem 4.2 we obtain the result.

Theorem 4.4. *Suppose $m, n \in \mathbb{N}, n > m \geq 7$ and $m-1 \nmid (n-3)$. Then*

$$\begin{aligned} r(K_{1,m-1}, T_n^*) &= m+n-4, \\ r(T'_m, T_{m+3}^*) &= r(T_m^*, T_{m+3}^*) = 2m-1, \\ r(T'_m, T_n^*) &= m+n-4 \text{ or } m+n-5, \\ m+n-6 &\leq r(T_m^*, T_n^*) \leq m+n-4. \end{aligned}$$

Proof. From Lemma 4.1 we have $r(T_m, T_n^*) \leq m+n-4$ for $T_m \in \{K_{1,m-1}, T'_m, T_m^*\}$. By Lemma 2.3, $r(K_{1,m-1}, T_n^*) \geq m-1+n-3$, $r(T'_m, T_{m+3}^*) \geq 2m-1$, $r(T_m^*, T_{m+3}^*) \geq 2m-1$, $r(T'_m, T_n^*) \geq m-2+n-3$ and $r(T_m^*, T_n^*) \geq m-3+n-3$. Thus the theorem is proved.

Theorem 4.5. Suppose $m, n \in \mathbb{N}$, $n > m \geq 7$, $n = k(m-1) + b$, $k \in \mathbb{N}$, $b \in \{0, 1, \dots, m-2\}$, $b \neq 3$ and $\frac{m-b}{2} \leq k \leq m+2-b$. Let G_m be a connected graph of order m such that $ex(m+n-4; G_m) \leq \frac{1}{2}(m-2)(m+n-4)$ or $G_m \in \{P_m, T_m^*\}$. Then $r(G_m, T_n^*) = m+n-4$ or $m+n-5$.

Proof. By Lemma 4.1 we only need to show that $r(G_m, T_n^*) > m+n-6$. Set $G = (2k+b-m)K_{m-2} \cup (m+2-b-k)K_{m-3}$. Then $|V(G)| = (2k+b-m)(m-2) + (m+2-b-k)(m-3) = m+n-6$. We also have $\Delta(G) \leq m-2$ and $\Delta(\overline{G}) \leq m+n-6-(m-3) = n-3$. Now it is clear that G_m is not a subgraph of G and that T_n^* is not a subgraph of \overline{G} . So $r(G_m, T_n^*) > |V(G)|$, which completes the proof.

Remark 4.1 If $p \geq m \geq 6$ and T_m is a tree on m vertices with a vertex adjacent to at least $\lfloor \frac{m-1}{2} \rfloor$ vertices of degree 1, in [9] Sidorenko proved that $ex(p; T_m) \leq \frac{(m-2)p}{2}$. Thus, G_m can be replaced with T_m in Lemma 4.1, Theorems 4.1, 4.2, 4.3 and 4.5.

5. The Ramsey number $r(G_m, T'_n)$ for $m < n$.

Theorem 5.1. Let $m, n \in \mathbb{N}$, $n > m \geq 6$ and $m-1 \mid (n-3)$. Suppose that G_m is a connected graph of order m satisfying $ex(m+n-3; G_m) \leq \frac{(m-2)(m+n-3)+m+n-4}{2}$ or $G_m \in \{T_m^*, P_m\}$. Then $r(G_m, T'_n) = m+n-3$.

Proof. By Lemma 2.9 we may assume $ex(m+n-3; G_m) \leq \frac{(m-2)(m+n-3)}{2}$. Suppose $n-3 = k(m-1)$ and $G = (k+1)K_{m-1}$. Then $|V(G)| = m+n-4$ and $\Delta(\overline{G}) = n-3$. Clearly, G_m is not a subgraph of G and T'_n is not a subgraph of \overline{G} . Thus $r(G_m, T'_n) > m+n-4$. Since $m-1 \mid (n-3)$, we have $n \geq m+2$ and so $4 \leq m-2 \leq n-4$. Hence, using Lemma 2.5 we see that $ex(m+n-3; T'_n) = \lfloor \frac{(n-2)(m+n-4)-(m-1)}{2} \rfloor < \frac{(n-2)(m+n-3)-(m+n-4)}{2}$. Therefore

$$ex(m+n-3; G_m) + ex(m+n-3; T'_n) < \binom{m+n-3}{2}.$$

Applying Lemma 2.1 we see that $r(G_m, T'_n) \leq m+n-3$. So the result follows.

Lemma 5.1. Let $m, n \in \mathbb{N}$, $n > m \geq 6$ and $m-1 \nmid (n-3)$. Suppose that G_m is a connected graph of order m satisfying $ex(m+n-4; G_m) < \frac{(m-2)(m+n-4)}{2}$ or $G_m \in \{T_m^*, P_m\}$. Then $r(G_m, T'_n) \leq m+n-4$.

Proof. Since $m-1 \nmid (n-3)$ we have $m-1 \nmid (m+n-4)$. Thus applying Lemma 2.9 we have $ex(m+n-4; T_m^*) \leq \frac{(m-2)(m+n-5)}{2}$ and $ex(m+n-4; P_m) \leq \frac{(m-2)(m+n-5)}{2}$. As $n > m$ we have $3 \leq m-3 \leq n-4$. By Lemma 2.5, $ex(m+n-4; T'_n) = \lfloor \frac{(n-2)(m+n-5)-(m-2)}{2} \rfloor \leq \frac{(n-2)(m+n-5)-(m-2)}{2}$. Thus

$$ex(m+n-4; G_m) + ex(m+n-4; T'_n) < \frac{(m-2+n-2)(m+n-5)}{2} = \binom{m+n-4}{2}.$$

This together with Lemma 2.1 yields the result.

Theorem 5.2. *Let $m, n \in \mathbb{N}$, $n > m \geq 6$ and $m - 1 \nmid (n - 3)$. Then $r(T_m^*, T'_{m+1}) = 2m - 3$ and $r(T_m^*, T'_n) = m + n - 4$ or $m + n - 5$ for $n \geq m + 3$. Suppose that G_m is a connected graph of order m satisfying $ex(m + n - 4; G_m) < \frac{(m-2)(m+n-4)}{2}$ or $G_m \in \{T_m^*, P_m\}$. If $m + n - 5 = (m - 1)x + (m - 2)y$ for some nonnegative integers x and y , then $r(G_m, T'_n) = m + n - 4$.*

Proof. By Lemma 2.3, $r(T_m^*, T'_{m+1}) \geq 2(m - 1) - 1 = 2m - 3$ and $r(T_m^*, T'_n) \geq m - 3 + n - 2$ for $n \geq m + 3$. By Lemma 5.1, $r(G_m, T'_n) \leq m + n - 4$. Thus, $r(T_m^*, T'_{m+1}) = 2m - 3$. Now applying Lemma 2.10 we deduce the remaining result.

From Theorem 5.2 and the proof of Theorem 4.3 we deduce the following result.

Theorem 5.3. *Suppose $m, n \in \mathbb{N}$, $n > m \geq 6$, $n = k(m - 1) + b = q(m - 2) + a$, $k, q \in \mathbb{N}$, $a \in \{0, 1, \dots, m - 3\}$ and $b \in \{0, 1, \dots, m - 2\} - \{3\}$. Let G_m be a connected graph of order m such that $ex(m + n - 4; G_m) < \frac{(m-2)(m+n-4)}{2}$ or $G_m \in \{P_m, T_m^*\}$. If one of the following conditions*

- (i) $b \in \{1, 2, 4\}$,
- (ii) $b = 0$ and $k \geq 3$,
- (iii) $n \geq (m - 3)^2 + 2$,
- (iv) $n \geq m^2 - 1 - b(m - 2)$,
- (v) $a \geq 3$ and $n \geq (a - 4)(m - 1) + 4$

holds, then $r(G_m, T'_n) = m + n - 4$.

6. The Ramsey number $r(T_m, K_{1,n-1})$ for $m < n$.

The following two propositions are known.

Proposition 6.1 (Burr[1]). *Let $m, n \in \mathbb{N}$ with $m \geq 3$ and $m - 1 \mid (n - 2)$. Let T_m be a tree on m vertices. Then $r(T_m, K_{1,n-1}) = m + n - 2$.*

Proposition 6.2 (Guo and Volkmann [6, Theorem 3.1]). *Let $m, n \in \mathbb{N}$, $m \geq 3$ and $n = k(m - 1) + b$ with $k \in \mathbb{N}$ and $b \in \{0, 1, \dots, m - 2\} - \{2\}$. Let $T_m \neq K_{1,m-1}$ be a tree on m vertices. Then $r(T_m, K_{1,n-1}) \leq m + n - 3$. Moreover, if $k \geq m - b$, then $r(T_m, K_{1,n-1}) = m + n - 3$.*

Theorem 6.1. *Let $m, n \in \mathbb{N}$, $n \geq m \geq 3$, $m - 1 \nmid (n - 2)$, $n = q(m - 2) + a$, $q \in \mathbb{N}$ and $a \in \{2, 3, \dots, m - 3\}$. Let $T_m \neq K_{1,m-1}$ be a tree on m vertices. If $n \geq (a - 3)(m - 1) + 3$, then $r(T_m, K_{1,n-1}) = m + n - 3$.*

Proof. Since $q(m - 2) = n - a \geq (a - 3)(m - 2)$ we have $q \geq a - 3$. Set $G = (a - 2)K_{m-1} \cup (q - (a - 3))K_{m-2}$. Then $|V(G)| = (a - 2)(m - 1) + (q - (a - 3))(m - 2) = m + n - 4$ and $\Delta(\overline{G}) \leq n - 2$. Clearly, T_m is not a subgraph of G and $K_{1,n-1}$ is not a subgraph of \overline{G} . Thus $r(T_m, K_{1,n-1}) > |V(G)| = m + n - 4$. By Proposition 6.2, $r(T_m, K_{1,n-1}) \leq m + n - 3$. So $r(T_m, K_{1,n-1}) = m + n - 3$. This proves the theorem.

Theorem 6.2. *Let $m, n \in \mathbb{N}$ with $n > m \geq 5$ and $m - 1 \nmid (n - 2)$. Then $r(T_m^*, K_{1, n-1}) = m + n - 3$ or $m + n - 4$. Moreover, if $m + n - 4 = (m - 1)x + (m - 2)y + 2(m - 3)z$ for some nonnegative integers x, y and z , then $r(T_m^*, K_{1, n-1}) = m + n - 3$.*

Proof. By Proposition 6.2, $r(T_m^*, K_{1, n-1}) \leq m + n - 3$. By Lemma 2.3 we have $r(T_m^*, K_{1, n-1}) \geq m + n - 4$. If $m + n - 4 = (m - 1)x + (m - 2)y + 2(m - 3)z$ for some nonnegative integers x, y and z , setting $G = xK_{m-1} \cup yK_{m-2} \cup zK_{m-3, m-3}$ we find $\Delta(\overline{G}) \leq n - 2$. Clearly, G does not contain any copies of T_m^* , and \overline{G} does not contain any copies of $K_{1, n-1}$. Thus, $r(T_m^*, K_{1, n-1}) > |V(G)| = m + n - 4$ and so $r(T_m^*, K_{1, n-1}) = m + n - 3$. This proves the theorem.

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